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Models of the Term Structure of Interest Rates: Review, Trends, and Perspectives

Oldrich Alfons Vasicek[1] - Vasicek Associates, U.S.A.
Francisco Venegas-Martínez[2] - Instituto Politécnico Nacional, México

Abstract

This paper provides an overview of models of the term structure of interest rates. It is a technical exposition of the theory of arbitrage-free behavior of interest rates of different maturities, requiring no prior knowledge of the field. The models acquire current relevance for their ability to describe and explain negative interest rates, as they have been observed in Europe and Asia. In an environment of uncertainty generated by a global economic recession, current economic conditions affect the behavior of interest rates, which invites a more detailed review of the factors that influence their dynamics. This article aims to review the trends and perspectives of models of the term structure and highlight future research areas.
JEL classification: D50, E43.
Keywords: interest rates, term structure, Vasicek model.

Modelos de la estructura de plazos de las tasas de interés: Revisión, tendencias y perspectivas

Resumen

El trabajo proporciona una descripción general de los modelos de estructuras de plazos de las tasas de interés. Se trata de un planteamiento técnico de la teoría del comportamiento libre de arbitraje de tasas de interés de distintos vencimientos. Los modelos de tasa corta están ganando relevancia en la actualidad por su capacidad para describir y explicar la existencia de tasas de interés negativas como se ha observado en Europa y Asia. Las condiciones económicas actuales, en un entorno de incertidumbre generado por una recesión económica global, afectan el comportamiento de las tasas de interés, lo cual invita a realizar una revisión más cuidadosa de los factores que influyen en la dinámica de las mismas. Este artículo tiene como objetivo revisar las tendencias y perspectivas de los modelos de estructuras de plazos y destacar algunas áreas para futuras investigaciones.
Clasificación JEL: D50, E43.
Palabras clave: tasas de interés, estructuras de plazos, modelo de Vasicek.

[1] Email. oldrich.vasicek@gmail.com
[2] Corresponding author. Email. fvenegas1111@yahoo.com.mx
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1. Introduction

This paper provides an overview of interest rate term structure models. It is a technical exposition of the theory of arbitrage-free behavior of interest rates of different maturities. It is sufficiently comprehensive and no prior knowledge of the field is assumed.

These models are gaining relevance today due to their ability to describe and explain the existence of negative interest rates, as it has been observed in Europe and Asia. In an environment of uncertainty generated by a global economic recession intensified in its depth, speed, and scope by the COVID-19 pandemic, current economic conditions affect interest rates´ behavior. This research aims to review the trends and perspectives of interest rate term structure models and highlight future research areas.

This research is organized as follows: Section 2 provides detailed information on the evolution of the theory and the models of term structure that have been proposed in the literature; Section 3 presents an in-depth analysis of the Heath, Jarrow, and Morton (1992) model, which is a frame of reference in the theory of term structures; Section 4 establishes some proposals for future research; finally, Section 5 provides the conclusions.

2. Theory of term structures and yield curves

Interest rates are a function of time and term (time to maturity). As a function of time, rates behave as stochastic processes. As a function of term, interest rates on a given date constitute the *term structure*, also called the *yield curve*. Term structure models describe the behavior in time of interest rates of different maturities as a joint stochastic process.

Term structure models are a necessary tool for valuation and risk management of interest rate contingent claims—that is, securities or transactions whose payoff depends on future values of interest rates. Examples are callable bonds (callable or redeemable, the issuer's right to buy back the bond from the holder), putable bonds (the holder's right to return the bond to the issuer), swaps, swaptions (the holder's right to enter into a swap), ceilings and floors, etc. For instance, a bond will be called if its value on the call date is greater than the call price. To determine the current value of the bond, it is necessary to know the subsequent behavior of interest rates. The same is true for all debt securities subject to prepayment, such as mortgages with refinancing options.

The immediate acceptance and application of term structure models in banking and investment practice was due to the fact that there are few financial instruments whose value is not in some degree dependent on future interest rates. Even stock options such as calls and puts depend on the development of interest rates. Interest rate models enter into valuation of firms and their liabilities. Besides valuation, term structure models are necessary for interest rate risk measurement, management and hedging.[3]

[3] See, for example, Venegas-Martínez (2002) and (2003), as well as Venegas-Martínez and González-Aréchiga (2002).

Interest rates of different maturities behave as a joint stochastic process. Not all joint processes, however, can describe interest rate behavior in an efficient market. For instance, suppose that a term structure model postulates that rates of all maturities change in time by equal amounts, that is, that yield curves move by parallel shifts (which, empirically, appears to be a reasonable first-order approximation). It can be shown that in this case a portfolio consisting of a long bond and a short bond would always outperform a medium-term bond with the same Macaulay duration. In an efficient market, supply and demand would drive the price of the medium maturity bond down and the prices of the long and short bonds up. As this would cause the yield on the medium bond to increase and the yields on the long and short bonds to decrease, the yield curves would not stay parallel. This model therefore cannot describe interest rate behavior.

In order that riskless arbitrage opportunities are absent, the joint process of interest rate behavior must satisfy some conditions. Determining these conditions and finding processes that satisfy them is the purpose of *term structure theory*. Term structure *models* are specific applications of term structure theory.

The joint stochastic process is driven by a number of sources of uncertainty. For continuous processes, the sources of uncertainty are often specified as Wiener processes. If the evolution of the yield curve can be represented by Markovian state variables, these variables are called *factors*.

2.1 Some necessary definitions

All rates here are annualized continuously compounded rates. Time is measured in years. A unit loan of term S at a fixed continuously compounded rate r pays at maturity the amount $\exp(rS)$. A rate R_n compounding at annual frequency n (such as n = 12 for monthly compounded rate) relates to the continuously compounded rate r by

$$1+R_n/n=e^{r/n} \tag{1}$$

Let $B(t,T)$ be the price at time t of a default-free zero-coupon bond maturing at time T with unit maturity value. Yield to maturity $R(t,S)$ at time t with term $S = T - t$ is defined as the continuously compounded rate of return on the bond,

$$R(t,S)=-\frac{1}{S}\log B(t,t+S)\,. \tag{2}$$

The instantaneous interest rate $r(t)$ will be called the *short rate*,

$$r(t)=\lim_{S\to 0}R(t,S) \tag{3}$$

An asset that accrues interest at a short rate is called a *money market account*,

$$M(t) = \exp\left(\int_0^t r(\tau)\,\mathrm{d}\,\tau \right). \tag{4}$$

Forward rates $f(t,T)$ are defined by the equation

$$B(t,T) = \exp\left(-\int_t^T f(t,\tau)\,\mathrm{d}\,\tau \right). \tag{5}$$

or equivalently,

$$f(t,T) = -\frac{\partial}{\partial T}\log B(t,T)\,. \tag{6}$$

Forward rates are the marginal rates of return from committing a bond investment for an additional instant. The forward rate for the current date is the short rate,

$$f(t,t) = r(t)$$

Most bonds pay coupons during their terms. A coupon bond is just a package of discount bonds, one for each coupon or principal payment.

2.2 One-factor models

A general theory of one-factor term structure models was given by Vasicek (1977). He assumed that

(A1) The short rate follows a continuous Markovian process.
(A2) The price $B(t,T)$ of a bond is determined by the assessment at time t of the segment $\{r(\tau), t \le \tau \le T\}$ of the short rate process over the term of the bond.
(A3) The market is efficient; that is, there are no transaction costs, information is available to all investors simultaneously, and every investor acts rationally (prefers more wealth to less, and uses all available information).

Assumption (A3) implies that investors have homogeneous expectations and that no profitable arbitrage without risk is possible. By assumption (A1), the development of the short rate on an interval $[t,T]$, $t \le T$, given its values prior to time t, depends only on the current value $r(t)$. Assumption (A2) then implies that the price $B(t,T)$ is a function of $r(t)$. Thus, the value of the short rate is the only state variable for the whole term structure. Let the dynamics of the short rate be given by

$$\mathrm{d}\,r(t) = \zeta(r,t)\,\mathrm{d}\,t + \varphi(r,t)\,\mathrm{d}\,W(t) \tag{7}$$

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where $W(t)$ is a Wiener process (standard Brownian motion). Denote the mean and variance of the instantaneous rate of return of the bond with price $B(t,T)$ by $\mu(t,T)$ and $\sigma^2(t,T)$, respectively,

$$\frac{\mathrm{d}\,B(t,T)}{B(t,T)} = \mu(t,T)\,\mathrm{d}\,t - \sigma(t,T)\,\mathrm{d}W(t)\,. \tag{8}$$

The negative sign in volatility $\sigma(t,T)$ does not pose a problem since $W(t)$ and $-W(t)$ have the same distribution. Consider an investor who at time t issues w_1 units of a bond with maturity date T_1 and simultaneously buys w_2 units of a bond with maturity date T_2. Suppose the quantities w_1 and w_2 are chosen to be proportional to $\sigma(t,T_2)$ and $\sigma(t,T_1)$, respectively. Then the position is instantly riskless and should realize the short rate of return $r(t)$. It follows that the ratio $(\mu(t,T) - r(t)) / \sigma(t,T)$ is independent of T. Its common value $\lambda(t)$ is called the *market price of risk*, as it specifies the increase in the expected rate of return on a bond per an additional unit of risk. We thus have

$$\mu(t,T) = r(t) + \lambda(t)\sigma(t,T) \tag{9}$$

By applying Itô's lemma to the price $B = B(t,T,r)$ and comparing the result with equation (8), taking into account (9), it is obtained that

$$\frac{\partial B}{\partial t} + (\zeta + \varphi\lambda)\frac{\partial B}{\partial r} + \tfrac{1}{2}\varphi^2\frac{\partial^2 B}{\partial r^2} - rB = 0 \tag{10}$$

This is a second-order partial differential equation. The bond price is subject to the boundary condition $B(T,T) = 1$.

The solution of (10) is given by:

$$B(t,T) = \mathrm{E}_t \exp\left(-\int_t^T r(\tau)\,\mathrm{d}\,\tau - \tfrac{1}{2}\int_t^T \lambda^2(\tau)\,\mathrm{d}\tau + \int_t^T \lambda(\tau)\,\mathrm{d}W(\tau)\right) \tag{11}$$

This equation, called the *fundamental bond pricing equation* (the *Vasicek equation*), fully describes the term structure and its behavior.

The bond pricing equation was initially derived as the solution to a partial differential equation under certain assumptions, but it is valid generally for *any* arbitrage-free term structure model. The equation is valid even in the case of multiple factors or multiple risk sources, if the products in the equation are interpreted as inner products of vectors.

Every arbitrage-free term structure model is either a direct application of that equation, or it assumes that the equation is true for bonds and uses it to price interest rate derivatives (as in the Heath, Jarrow, Morton model).

2.3 The Vasicek model

Vasicek (1977) provides an example of a term structure model in which the short rate follows a mean reverting random walk (the Ornstein-Uhlenbeck process)

$$\mathrm{d}r = \kappa(\theta - r)\,\mathrm{d}t + \varphi\,\mathrm{d}W \tag{12}$$

and the market price of risk $\lambda(t,r) = \lambda$ is constant. The Ornstein-Uhlenbeck process is a Markov process with normally distributed increments and a stationary distribution. The instantaneous drift $\kappa(\theta - r)$ represents a force that keeps pulling the process towards its long-term mean θ with magnitude proportional to the deviation of the process from the mean. The constant κ represents the speed of the adjustment. The stochastic element $\varphi \mathrm{d}W$ causes the process to fluctuate around the level θ in a random, but continuous, fashion. The conditional expectation and variance of the process, given the current level, are

$$\mathrm{E}_t\, r(T) = \theta + (r(t) - \theta)e^{-\kappa(T-t)} \tag{13}$$

and

$$\mathrm{Var}_t r(T) = \frac{\varphi^2}{2\kappa}\left(1 - e^{-2\kappa(T-t)}\right), \tag{14}$$

respectively.

The expectation in (11) can be evaluated, or Eq. (10) solved, to produce the bond prices,

$$B(t,T) = \exp(D(t,T)(R(\infty) - r) - (T-t)R(\infty) - \frac{\varphi^2}{4\kappa}D^2(t,T)) \tag{15}$$

where

$$D(t,T) = \frac{1}{\kappa}(1 - e^{-\kappa(T-t)}) \tag{16}$$

and

$$R(\infty) = \theta + \lambda\varphi/\kappa - \tfrac{1}{2}\varphi^2/\kappa^2\,. \tag{17}$$

The function $D(t,T)$ describes the bond price's exposure to the stochastic factor $r(t)$. The mean $\mu(t,T)$ and standard deviation $\sigma(t,T)$ of the instantaneous rate of return on a bond maturing at time T are

$$\begin{aligned}\mu(t,T) &= r(t) + \lambda\varphi D(t,T)),\\ \sigma(t,T) &= \varphi D(t,T).\end{aligned} \tag{18}$$

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The longer the bond's term, the higher the variance of its instantaneous rate of return. The expected return in excess of the spot rate is proportional to the standard deviation. For a very long maturity (that is, as $T \to \infty$), the mean and standard deviation approach the limits

$$\mu(t,\infty) = r(t) + \lambda\varphi/\kappa,$$
$$\sigma(t,\infty) = \varphi/\kappa. \tag{19}$$

The term structure of interest rates is then calculated from (2) and (15) as

$$R(t, S) = R(\infty) + (r(t) - R(\infty))\frac{1}{S} D(t, t+S) + \frac{\varphi^2}{4\kappa S} D^2(t, t+S)\,. \tag{20}$$

Note that the yield of a very long bond, with $S \to \infty$, is $R(\infty)$, which explains the notation in (17). The yield curves given by (20) start at the current level $r(t)$ of the spot rate for $S = 0$ and approach a common asymptote $R(\infty)$ when $S \to \infty$. Depending on the value of $r(t)$, the yield curve can be monotonously increasing, a humped curve, or monotonously decreasing.

Interest rates are Gaussian. The advantage of the Vasicek model is its tractability. A feature of the model is that the interest rates can become negative. This property, rather than being a drawback, provides means to describe and explain this phenomenon.

2.4 Examples of term structure models

Various specific cases of term structure models have been proposed in the literature. For example, Cox, Ingersoll and Ross (1985b) obtain a model in which the short rate follows a process of the form

$$\mathrm{d}r = \kappa(\theta - r)\,\mathrm{d}t + \varphi\sqrt{r}\,\mathrm{d}W\,. \tag{21}$$

For this model, the market price of risk is given by $\lambda(t, r) = \eta\sqrt{r}$. In this case, bond prices can also be explicitly given. They have the form

$$B(t,T) = A(t,T)\exp(-D(t,T)r(t)) \tag{22}$$

The quantity $D(t,T)$, which measures the degree of exposure of bond prices to the stochastic factor $r(t)$, is given by

$$D(t,T) = \frac{1 - e^{-\gamma(T-t)}}{\gamma + \frac{1}{2}(\kappa - \varphi\eta - \gamma)(1 - e^{-\gamma(T-t)})} \tag{23}$$

where

$$\gamma = (2\varphi^2 + (\kappa - \varphi\eta)^2)^{1/2}\,. \tag{24}$$

In this case, interest rates are always non-negative, with a non-central Chi-square distribution. The Cox-Ingersoll-Ross (CIR) model is derived from general equilibrium (see Cox, Ingersoll, Ross, 1985a). In contrast, most other models are obtained by the no-arbitrage argument.

Note that Cox, Ingersoll, Ross in their 1985 paper use the notation $\lambda = -\varphi\eta$, which gives the wrong impression about the sign of the market price of risk. For a positive bond risk premium $E[dB/B] - r = \eta\sqrt{r}\,E[dW dB/B]$, the quantity λ in the CIR model must be negative.

Some restrictions must be placed on the model coefficients to avoid singular behaviors. For example, the process $r = r(t)$ will have negative mean reversion (repulsion from the mean) under the martingale measure if $\kappa - \varphi\eta < 0$. When $\kappa - \varphi\eta < -\varphi\sqrt{2}$, the expected balance in a money market account will become infinite in a finite time.

Hull and White (1990) extended the Vasicek and CIR models by allowing the parameters in (12) and (21), as well as the market price of risk, to be time dependent. This has the advantage that the model can be made consistent with initial data. For example, by making θ a function of time, the model can be made to fit exactly the initial term structure of interest rates (which is not possible with time-homogeneous models). Similarly, making the volatility φ a function of time allows the model to be calibrated to the term structure of swaption volatilities.

Hull and White offer closed-form solutions for bond prices that they call the *extended Vasicek* and *extended Cox, Ingersoll, Ross* models. These cases belong to the class of models that Duffie and Kan (1994) call the *affine term structure models*, in which bond prices have the form (22).

Black, Derman and Toy (1990) and Black and Karasinski (1991) give a model with

$$d\log r = \kappa(t)(\log\theta(t) - \log r)\,dt + \varphi(t)\,dW\,. \tag{25}$$

In this case, bond prices cannot be given in closed form equation, but can be calculated numerically. Interest rates are lognormal.

Unlike other models, the lognormal models are not consistent with market equilibrium. There is no economy in equilibrium in which interest rates can be described by these models. This can be proven formally, but an indication of the problems that lognormal models have is that the expected future balance (even in a short moment!) in the money market account is infinite,

$$E\frac{M(T)}{M(t)} = \infty, \quad t < T \tag{26}$$

Lognormal models produce infinite Eurodollar futures prices.

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2.5 Classification of term structure models

To classify the numerous term structure models proposed in the literature, it is possible to look at a number of aspects:

1. Number of risk sources/number of factors:
   One factor models. The factor is typically the short rate.
   Two factors (Brennan/Schwartz 1979, Longstaff/Schwartz 1992).
   Multiple factors (Langetieg 1980).
   Models with no factors (Heath/Jarrow/Morton 1992).

Factors are state variables that contain all available information. Factors are Markov processes such that rates and prices are functions of the factors only.

Factor model are typically constructed in the following way:

Identify the stochastic factors and describe their behavior.
Specify pricing of risk..
Solve partial differential equation for bond prices, or calculate the expectation in (11).

Hedging interest rate risk is possible in factor models.

2. Nature of processes:
   Mean reverting vs. random walk.
   Normal vs. Lognormal vs. Square-root processes.
   Jump processes, etc.

3. Initial term structure:
   Implied yield curve (*normative models*). These time-homogeneous models specify a family of curves that can be attained by the term structure (for instance, Vasicek 1977, Cox/Ingersoll/Ross 1985, Longstaff/Schwartz 1990). The initial yield curve is assumed to belong to that family. The obvious disadvantage is that the model does not exactly agree with the current bond pricing. The difference between the actual and implied yield is attributed to mispricing, or to bond-specific characteristics, such as liquidity. The advantage is that the implied future yield curves are realistic.

   Fitted yield curve (*descriptive models*). These models take initial yield curve for given (for instance, Hull/White 1990, Ho/Lee 1986, Heath/Jarrow/Morton 1992). The models fit the current bond pricing exactly. These models are by nature non-time-homogeneous. The drawback is that the model implies certain families of possible future yield curves, and the user knows today that tomorrow he/she will be using a yield curve that may not belong to that family, violating his/her own assumptions. This is illustrated on an example in Section 3.7.

Multifactor models: Normative constant coefficient models with sufficient number of factors (e.g., benchmark yields). Such models (for instance, El Karoui and Lacoste 1992, Duffie and Kan 1994, Vasicek 2020), while computationally complex, have the advantages of both the normative and descriptive models without the disadvantages of either.

Table 1 provides a few examples of one-factor models and their properties. The list is not intended to be exhaustive.

Table 1. Some one-factor models and their properties.

Model	**Vasicek**	**Hull/White**	**Cox/Ingersoll/Ross**	**Black/Derman/Toy**	**Ho/Le**
Mean reversion	Linear	Linear	Linear	Nonlinear	None
Volatility	φ	φ	$\varphi\sqrt{r}$	φr	φ
Constant parameters	Yes	No	Yes	No	No
Non-negative interest rates	No	No	Yes	Yes	No
Interest rate distribution	Normal	Normal	Non-central chi square	Lognormal	Norm
Closed-form solution	Yes	Yes	Yes	No	Yes

Source: Authors' elaboration.

Single-factor models are often analytically tractable, but it appears that in reality interest rates are driven by multiple sources of uncertainty, which requires more than one factor. The assumption of constant volatility is possibly also unrealistic. Finally, it should be noted that what was previously seen as a weakness in several of the models, namely the generation of negative short rates, is an advantage since they allow describing and explaining the existence of negative interest rates that have been observed in several regions of the world.

2.6 Contingent claim pricing

One of the main tasks of term structure models in applications is pricing of *interest rate contingent claims* (interest rate derivatives). This could be approached in several ways. For one-factor models it can be shown, by means of an arbitrage argument similar to that above for bonds, that the price $P(t)$ of any interest rate derivative satisfies the partial differential equation (10). The valuation of the

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claim is then accomplished by solving that equation subject to boundary conditions that describe the derivative asset payouts.

For example, the boundary conditions for a zero-coupon bond maturing at time T_M with par value F, callable after time T_C at call price C are:

$$P(T_M, r) = F, \quad P(t,r) \leq C \text{ for } T_C \leq t \leq T_M .$$

If a closed-form solution cannot be obtained, then the solution can be approximated numerically by a tree or a finite-difference lattice.

A more general method is to realize that such a solution satisfies the equation

$$P(t) = \mathrm{E}_t\, P(T) \exp\left(-\int_t^T r(\tau)\,\mathrm{d}\,\tau - \tfrac{1}{2}\int_t^T \lambda^2(\tau)\,\mathrm{d}\tau + \int_t^T \lambda(\tau)\,\mathrm{d}\,W(\tau) \right) \tag{27}$$

over any interval (t,T) in which the asset makes no payments. This equation is valid even in cases where there are no Markovian state variables. However, calculating the expectation can be more complex than solving a partial differential equation.

2.7 Liquidity premium

The difference between the forward rate and the expected spot rate has traditionally been called the *liquidity premium*. Let

$$\varphi(t,T) = \frac{\partial \sigma(t,T)}{\partial T} \tag{28}$$

be the volatility of the forward interest rate $f(t,T)$. The liquidity premium (or term premium, as it should be called) $\pi(t,T)$ is given by

$$\pi(t,T) = f(t,T) - \mathrm{E}_t\, r(T) = \mathrm{E}_t \int_t^T \varphi(\tau,T)\lambda(\tau)\,\mathrm{d}\,\tau - \mathrm{E}_t \int_t^T \varphi(\tau,T)\sigma(\tau,T)\,\mathrm{d}\,\tau \tag{29}$$

The liquidity premium in a term structure of interest rates has two components: The first component is driven by the market price of risk. It is equal to the expected aggregate of the market price of risk over the span of the forward rate, multiplied by the forward rate volatility. The second component is equal to the negative of the expected aggregate of the product of the bond price volatility times the forward rate volatility, over the forward rate span. This component, which is present even if the market price of risk is zero, arises as a result of the nonlinear relationship between prices and rates.

2.8 The martingale measure

The modern theory of derivative asset pricing (see Harrison and Kreps, 1979) introduces a change of probability measure as the basic pricing tool. There exists an equivalent probability measure **P***, called the *martingale measure*, or (incorrectly) the *risk-neutral measure*, such that the value $P(t)$ of any asset expressed in units of the money market account $M(t)$ follows a martingale under that measure,

$$\frac{P(t)}{M(t)} = \mathrm{E}_t^* \frac{P(T)}{M(T)}\,. \tag{30}$$

and therefore

$$P(t) = \mathrm{E}_t^*\, P(T) \exp\left(-\int_t^T r(\tau)\,\mathrm{d}\tau \right) \tag{31}$$

over any interval (t,T) in which the asset makes no payments. The process

$$W^*(t) = W(t) - \int_0^t \lambda(\tau)\,\mathrm{d}\,\tau \tag{32}$$

is a Wiener process under the martingale probability measure **P***. For each asset price $P(t)$, it follows that

$$\frac{\mathrm{d}P}{P} = (r + \sigma_P \lambda)\,\mathrm{d}t - \sigma_P\,\mathrm{d}W) = r\,\mathrm{d}t - \sigma_P\,\mathrm{d}W^* \tag{33}$$

The price of risk under the martingale measure is zero, and the expected rate of return for any security is the short rate $r(t)$. The theory states that financial assets are priced as the expected value of their future cashflows discounted to present at the short rate, the expectation being taken with respect to the martingale measure under which the expected rate of return on all assets is the short rate. Bond prices are given by

$$B(t,T) = \mathrm{E}_t^* \left[\exp\left\{ -\int_t^T r(\tau)\,\mathrm{d}\tau \right\} \middle|\, r(t) \right] \tag{34}$$

The Radon-Nikodym derivative of the new measure with respect to the actual measure is

$$\frac{\mathrm{d}\mathbf{P}^*}{\mathrm{d}\mathbf{P}} = \exp\left(-\tfrac{1}{2} \int_t^{T_{\max}} \lambda^2(\tau)\,\mathrm{d}\tau + \int_t^{T_{\max}} \lambda(\tau)\,\mathrm{d}W(\tau) \right) \tag{35}$$

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Since

$$\mathrm{E}^* X = \mathrm{E}\, X \frac{\mathrm{d}\mathbf{P}^*}{\mathrm{d}\mathbf{P}} \tag{36}$$

for any variable X, applying equations (34) and (35) to a bond price produces the fundamental bond valuation equation (11).

2.9 The Heath-Jarrow-Morton model

The Heath/Jarrow/Morton model (1992) is actually more than a model: it is a framework. While it is consistent with the fundamental bond pricing equation (11), it allows pricing interest rate derivatives without knowing the market price of risk $\lambda(t)$. The price of risk is implicitly obtained from the current bond prices. This approach was proposed in essence by Ho and Lee (1986) and later formalized by Heath, Jarrow and Morton. Knowledge of the initial term structure $f(0,T)$, $T \geq 0$ and of the forward rate volatilities is sufficient for pricing interest rate dependent securities.

The model assumes that the current prices of bonds of all maturities are known, and uses the initial yield curve to price interest rate derivatives. By writing the dynamics of forward rates directly in terms of the process $W^*(t)$, it is possible to price interest rate contingent claims without explicitly specifying the price of risk. Forward rate volatilities are a model input. The model fits current pricing of bonds.

The model derives the dynamics of the forward rates as

$$f(t,T) - f(0,T) = \int_0^t \varphi(\tau,T)\sigma(\tau,T)\,\mathrm{d}\tau + \int_0^t \varphi(\tau,T)\,\mathrm{d}W^*(\tau) \tag{37}$$

where $\varphi(t,T)$ is the volatility of the forward rate $f(t,T)$ and

$$\sigma(\tau,T) = \int_\tau^T \varphi(\tau,s)\,\mathrm{d}s \tag{38}$$

is the bond volatility. If W^*, φ, and σ are vectors, their products are interpreted as inner products.

The relevance of equation (37) is that it determines the expected value under $\mathbf{P}^*$ of future forward rates, inferring it from the current prices and therefore implicitly containing the risk premium. The payouts of any interest rate contingent claim can be expressed in terms of future forward rates. The claim is priced as the discounted expected value of its payouts under the martingale measure for which $W^*(t)$ is a Wiener process.

In terms of bond prices, the model specifies that

$$B(t,T) = \frac{B(0,T)}{B(0,t)} exp\left(-\frac{1}{2}\int_0^t (\sigma^2(\tau,T) - \sigma^2(\tau,t))\, d\tau - \int_0^t (\sigma(\tau,T) - \sigma(\tau,t))\, dW^*(\tau)\right) \tag{39}$$

A three-line derivation of the Heath-Jarrow-Morton (1992) model is provided below. A detailed deduction will be made in section 3.

2.10 Derivation of the HJM model

From equations (8), (9), and (32), bond prices are subject to

$$\frac{\mathrm{d}\,B(\tau,T)}{B(\tau,T)} = r(\tau)\,\mathrm{d}\,\tau - \sigma(\tau,T)\,\mathrm{d}W^{*}(\tau) \tag{40}$$

Integrate equation (40) with respect to τ from 0 to t,

$$\log B(t,T) - \log B(0,T) = \int_0^t r(\tau)\,\mathrm{d}\,\tau - \tfrac{1}{2}\int_0^t \sigma^2(\tau,T)\,\mathrm{d}\,\tau + \int_0^t \sigma(\tau,T)\,\mathrm{d}W^{*}(\tau) \tag{41}$$

and differentiate equation (41) with respect to T. This produces equation (37).

For $T = t$,

$$-\log B(0,t) = \int_0^t r(\tau)\,\mathrm{d}\,\tau - \tfrac{1}{2}\int_0^t \sigma^2(\tau,t)\,\mathrm{d}\,\tau - \int_0^t \sigma(\tau,t)\,\mathrm{d}W^{*}(\tau) \tag{42}$$

Subtracting (42) from (41) yields (39).

2.11 Modeling asset prices in the HJM framework

Asset prices are subject to (33). From (39), the short rate is given by

$$r(t) = f(0,t) + \int_0^t \varphi(\tau,t)\sigma(\tau,t)\,\mathrm{d}\,\tau + \int_0^t \varphi(\tau,t)\,\mathrm{d}W^{*}(\tau) \tag{43}$$

and by integration,

$$\int_0^T r(\tau)\,\mathrm{d}\,\tau = \int_0^T f(0,\tau)\,\mathrm{d}\,\tau + \tfrac{1}{2}\int_0^T \sigma^2(\tau,T)\,\mathrm{d}\,\tau + \int_0^T \sigma(\tau,T)\,\mathrm{d}W^{*}(\tau) \tag{44}$$

An asset that has a single payout of $P(T)$ at time T is then priced at $t = 0$ as

$$P(0) = B(0,T)\,\mathrm{E}^{*}\,P(T)\exp\!\Big(-\tfrac{1}{2}\int_0^T \sigma^2(\tau,T)\,\mathrm{d}\,\tau - \int_0^T \sigma(\tau,T)\,\mathrm{d}W^{*}(\tau)\Big) \tag{45}$$

Suppose that rather just value the asset at present time t = 0, the analyst wants to model the price of the asset at time t. From (43)

$$\int_t^T r(\tau)\,\mathrm{d}\tau = \int_t^T f(0,\tau)\,\mathrm{d}\tau + \tfrac{1}{2}\int_0^t (\sigma^2(\tau,T) - \sigma^2(\tau,t))\,\mathrm{d}\tau + \tfrac{1}{2}\int_t^T \sigma^2(\tau,T)\,\mathrm{d}\tau$$
$$+\int_0^t (\sigma(\tau,T) - \sigma(\tau,t))\,\mathrm{d}W^*(\tau) + \int_t^T \sigma(\tau,T)\,\mathrm{d}W^*(\tau) \tag{46}$$

The price $P(t)$ of the asset is then given by

$$P(t) = \frac{B(0,T)}{B(0,t)} \exp(-\tfrac{1}{2}\int_0^t (\sigma^2(\tau,T) - \sigma^2(\tau,t))\,\mathrm{d}\tau - \int_0^t (\sigma(\tau,T) - \sigma(\tau,t))\,\mathrm{d}W^*(\tau))$$
$$= \mathrm{E}_t^* P(T) \exp(-\tfrac{1}{2}\int_t^T \sigma^2(\tau,T)\,\mathrm{d}\tau - \int_t^T \sigma(\tau,T)\,\mathrm{d}W^*(\tau)) \tag{47}$$

2.12 The yield curve

A graph of bond yields on a given date versus maturity is called a yield curve. Since yield quotes are usually available only for selected maturities (so-called benchmark yields), it has been necessary to fit a smooth curve to the discrete data. Substantial effort has been expended on yield curve interpolation (see, for instance, McCulloch 1971; Vasicek and Fong 1982; Nelson and Siegel 1987; Adams and van Deventer 1994). Although these methods work reasonably in practice, the resulting interpolations do not correspond to any arbitrage-free term structure model. There is no equilibrium term structure of interest rates under which the yield curves would form splines, whether cubic or quadratic or exponential. Vasicek (2020) proposed an interpolation of the yield curve that is compatible with an arbitrage-free model of the term structure of interest rates. The model is a multivariate time-homogeneous Gaussian yield factor model. In order for the interpolation method to result in stable yield curves, maximum stability interpolation is introduced as that which minimizes the integral of the yield variance over term.

3. Revisiting the Heath-Jarrow-Morton model

This section presents, in detail, the methodology developed by David Heath, Robert Jarrow, and Andrew Morton (HJM) in their article "*Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation*", published in 1992 in Econometrica, in which yield curves are generated based on the current forward rate (at all maturities) and an initial yield curve, as well as several examples of the HJM methodology.

To estimate the prices of a zero-coupon bond at different maturities, the Heath, Jarrow and Morton model begins with an exogenous specification of the stochastic dynamics of the forward rate and then endogenously determines, in a risk-neutral world, the stochastic dynamics of a zero-coupon bond.

The HJM methodology is similar to that of Ho and Lee (1986) and Hull and White (1990) in several ways. First, an initial yield curve is required, provided by the market at an earlier date. Likewise, the instantaneous forward rate trend is calibrated so that the standardized risk premium for volatility is zero. The differences with Ho and Lee (1986) and Hull and White (1990) are, basically: 1) the valuation process in HJM begins with an exogenous specification of the stochastic dynamics of the forward rate, 2) the hypothesis of expectations in HJM to value a bond is that the nominal is discounted with the average forward rate during the life of the instrument, which is why the price of the bond is a random variable, and 3) the calibration in HJM is an implicit procedure in the methodology, and it does not require adjustment arguments as in the case of Ho and Lee (1986) and Hull and White (1990). Under the HJM methodology, a bond's price is a random variable; the Monte Carlo method is a handy tool in practice. An advantage of the HJM methodology that should be highlighted is that it can be extended to various risk factors, for example, short- and long-term factors. However, a limitation of the HJM methodology is that negative forward rates can occur with positive probability.

It is worth mentioning that when starting from an exogenous specification for the stochastic dynamics of the short rate, the expectation hypothesis to value a bond is that, first, the nominal is discounted with the average of the short rate during the term of the security and, subsequently, the conditional expected value is taken from the information available on the placement date. From the Feynman-Kac Theorem, this is the only expectation hypothesis congruent with the partial differential equations approach. Thus, the HJM methodology is not compatible with such an approach.

3.1 Exogenous specification of the instantaneous forward rate

Consider a standard Brownian motion $(W(t))_{t\in[0,T]}$ defined over a fixed probability space with its augmented filtration $\left(\Omega, F, (F_t)_{t\in[0,T]}, \mathbf{P}\right)$. For the sake simplicity, it will be written $W(t) = W_t$ in what follows. In the HJM methodology, it is assumed that the dynamics of the forward rate, $f(t,T)$, is exogenously specified by the following stochastic differential equation[4]

$$\mathrm{d}f(t,T) = \alpha(t,T)\mathrm{dt} + \varphi(t,T)\mathrm{d}W_t, \tag{48}$$

where, according to (38), the functions $\alpha(t,T)$ and $\varphi(t,T)$ satisfy, almost surely with respect to $\mathbf{P}$, the following properties:

$$\int_0^T \left|\frac{\partial^k}{\partial T^k}\alpha(s,T)\right| \mathrm{d}s < \infty \qquad \text{and} \qquad \int_0^T \left|\frac{\partial^k}{\partial T^k}\varphi(s,T)\right|^2 \mathrm{d}s < \infty$$

[4] See, for example, Venegas-Martínez (2008).

for k = 0, 1. As usual, $\partial^0 \alpha(s,T)/\partial T^0 \equiv \alpha(s,T)$ and $\partial^0 \varphi(s,T)/\partial T^0 \equiv \varphi(s,T)$. Likewise, the price of a zero-coupon bond is assumed to be given by

$$B(t,T) = \exp\left\{-\int_t^T f(t,s)\mathrm{d}s\right\}, \tag{49}$$

which always defines the forward rate; the previous integral must remain finite. One of the tasks in this section is to endogenously determine the process associated with price, $B(t,T)$, which makes assumptions (48) and (49) consistent. Equation (42) may consider more than one uncertainty factor; for the moment, the subsequent analysis will consider only one factor.

3.2 Stochastic dynamics of the short rate

The task of this section is to determine the stochastic differential equation that leads to the short rate. Observe first that from (41), it follows

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)\mathrm{d}s + \int_0^t \varphi(s,T)\mathrm{d}W_s. \tag{50}$$

Therefore, the instantaneous rate satisfies

$$r_t = f(t,t) = f(0,t) + \int_0^t \alpha(s,t)\mathrm{d}s + \int_0^t \varphi(s,t)\mathrm{d}W_s. \tag{51}$$

In this manner,

$$\mathrm{E}[r_t|F_t] = f(0,t) + \int_0^t \alpha(s,t)\mathrm{d}s$$

and

$$\mathrm{Var}[r_t|F_t] = \int_0^t \varphi^2(s,t)\mathrm{d}s.$$

Also, note that the stochastic differential of the short rate is given by:

$$\mathrm{d}r_t = \frac{\partial f(0,t)}{\partial t}\mathrm{d}t + \frac{\partial}{\partial t}\left(\int_0^t \alpha(s,t)\mathrm{d}s\right)\mathrm{d}t + \frac{\partial}{\partial t}\left(\int_0^t \varphi(s,t)\mathrm{d}W_s\right)\mathrm{d}t. \tag{52}$$

The partial derivatives of the integrals on the right side of the above equation are calculated using Leibniz's rule, in such a way that

$$\frac{\partial}{\partial t}\left(\int_0^t \alpha(s,t)\mathrm{d}s\right)\mathrm{d}t = \left(\alpha(t,t) + \int_0^t \frac{\partial\alpha(s,t)}{\partial t}\mathrm{d}s\right)\mathrm{d}t$$

and

$$\frac{\partial}{\partial t}\left(\int_0^t \varphi(s,t)\mathrm{d}W_s\right)\mathrm{d}t = \varphi(t,t)\mathrm{d}W_t + \left(\int_0^t \frac{\partial\varphi(s,t)}{\partial t}\mathrm{d}W_s\right)\mathrm{d}t.$$

Consequently, equation (50) can be expressed as:

$$\mathrm{d}r_t = \left(\frac{\partial f(0,t)}{\partial t} + \alpha(t,t) + \int_0^t \frac{\partial\alpha(s,t)}{\partial t}\mathrm{d}s + \int_0^t \frac{\partial\varphi(s,t)}{\partial t}\mathrm{d}W_s\right)\mathrm{d}t + \varphi(t)\mathrm{d}W_t. \qquad (53)$$

where $\varphi(t) = \varphi(t,t)$ is the volatility of the short rate. This equation determines the behavior of the short rate. Notice that the trend of r_t is the slope of the initial forward rate. Obviously, due to the trend integrals in (53), the short rate evolution does not present the Markovian property. Once the dynamics that govern the behavior of r_t have been determined, given in equation (51), the dynamics of the price of the zero-coupon bond associated with r_t will be described in the subsequent sections.

3.3 Stochastic dynamics of the bond price

Given the exogenous specification of the stochastic dynamics of the instantaneous forward rate, the objective of this section is to endogenously determine the bond price, $B(t,T)$, that is consistent with assumptions (48) and (49). Let be

$$I_t = -\int_t^T f(t,s)\mathrm{d}s. \qquad (54)$$

In this case, Leibniz's rule produces the following result:

$$\mathrm{d}I_t = -\frac{\partial}{\partial t}\left(\int_t^T f(t,s)\mathrm{d}s\right)\mathrm{d}t = -\int_t^T \left(\frac{\partial f(t,s)}{\partial t}\mathrm{d}t\right)\mathrm{d}s + f(t,t)\mathrm{d}t \qquad (55)$$

The substitution of (48) in (55) and the fact that $f(t,t) = r_t$ leads to

$$\begin{aligned}\mathrm{d}I_t &= -\int_t^T \alpha(t,s)\mathrm{d}s\mathrm{d}t - \int_t^T \varphi(t,s)\mathrm{d}s\mathrm{d}W_t + r_t\mathrm{d}t \\ &= \left(r_t - \int_t^T \alpha(t,s)\mathrm{d}s\right)\mathrm{d}t - \left(\int_t^T \varphi(t,s)\mathrm{d}s\right)\mathrm{d}W_t. \qquad (56)\end{aligned}$$

If the trend and volatility of $\mathrm{d}I_t$ are denoted, respectively, by

$$U(t,T,r_t) = r_t - \int_t^T \alpha(t,s)\mathrm{d}s$$

and

$$V(t,T) = -\int_t^T \varphi(t,s)\mathrm{d}s = -\sigma(t,T)$$

it follows that

$$\mathrm{d}I_t = U(t,T,r_t)\mathrm{d}t + V(t,T)\mathrm{d}W_t. \tag{57}$$

Notice now that

$$B(t,T) = G(I_t) \qquad \text{with} \qquad G(I_t) = \exp\{I_t\}.$$

Consequently, Itô's lemma applied to G with respect to process (57) leads to

$$\begin{aligned}\mathrm{d}B &= \left(\frac{\partial G}{\partial I_t}U + (1/2)\frac{\partial^2 G}{\partial I_t^2}V^2\right)\mathrm{d}t + \frac{\partial G}{\partial I_t}V\mathrm{d}W_t \\ &= (GU + (1/2)GV^2)\mathrm{d}t + GV\mathrm{d}W_t \\ &= B(U + (1/2)V^2)\mathrm{d}t + BV\mathrm{d}W_t.\end{aligned}$$

Equivalently,

$$\begin{aligned}\mathrm{d}B(t,T) = &\left[r_t - \int_t^T \alpha(t,s)\mathrm{d}s + \frac{1}{2}\left(\int_t^T \varphi(t,s)\mathrm{d}s\right)^2\right]B(t,T)\mathrm{d}t \\ &-\left(\int_t^T \varphi(t,s)\mathrm{d}s\right)B(t,T)\mathrm{d}W_t.\end{aligned} \tag{58}$$

HJM's theoretical framework represented by equations (48), (52) and (58) fully describes the behavior of the instantaneous forward rate, the short rate, and the bond price. It is now necessary to move to the risk-neutral world to carry out the valuation process.

3.4 Risk neutral valuation in the HJM model

Consider a portfolio with two bonds with different maturities, T_1 and T_2. The value of the portfolio, at time t, with w_1 units of the bond maturing in T_1 and w_2 units of the bond maturing in T_2, is given by:

$$\Pi_t = w_1 B(t,T_1) + w_2 B(t,T_2). \tag{59}$$

The change in the value of the portfolio due to market fluctuations satisfies

$$\begin{aligned}\mathrm{d}\Pi_t = [&w_1\big(U(t,T_1,r_t) + (1/2)V^2(t,T_1)\big)B(t,T_1)\\ &+w_2(\,U(t,T_2,r_t) + (1/2)V^2(t,T_2))\,B(t,T_2)]\,\mathrm{d}t\\ &+[w_1V(t,T_1)B(t,T_1) + w_2V(t,T_2)B(t,T_2)]\mathrm{d}W_t.\end{aligned} \tag{60}$$

After choosing $w_1 = 1$ and

$$w_2 = -\frac{V(t,T_1)B(t,T_1)}{\mathrm{V}(t,T_2)B(t,T_2)}, \tag{61}$$

then, the coefficient of the term in $\mathrm{d}W_t$ is canceled, and, consequently, the portfolio is hedged against market risk. Therefore,

$$\mathrm{d}\Pi_t = \left(U(t,T_1,r_t) + (1/2)V^2(t,T_1) - \frac{U(t,T_2,r_t) + (1/2)V^2(t,T_2)}{V(t,T_2)}V(t,T_1)\right)B(t,T_1)\mathrm{d}t. \tag{62}$$

If, on the other hand, there is a credit market where agents can lend and borrow at the "spot" rate r_t, also called the short rate or the instantaneous rate (associated, in practice, with the smallest term interest rate available in the market), it follows that

$$\mathrm{d}\Pi_t = \Pi_t r_t \mathrm{d}t = \left(1 - \frac{V(t,T_1)}{V(t,T_2)}\right)r_t B(t,T_1)\mathrm{d}t. \tag{63}$$

After equating (62) with (63), it is obtained that

$$\frac{U(t,T_1,r_t) + \frac{1}{2}V^2(t,T_1) - r_t}{V(t,T_1)} = \frac{U(t,T_2,r_t) + \frac{1}{2}V^2(t,T_2) - r_t}{V(t,T_2)}. \tag{64}$$

The above ratios are independent of the expiration date. The left side of the equality in (64) only depends on T_1, and the right only depends on T_2. Therefore, it can be written

$$\lambda(r_t,t) = \frac{U(t,T,r_t) + \frac{1}{2}V^2(t,T) - r_t}{V(t,T)}. \tag{65}$$

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The function $\lambda(r_t, t)$ is the risk premium associated with the uncertainty factor $\mathrm{d}W_t$. In a risk-neutral world $\lambda(r_t, t) \equiv 0$.[5] Consequently, the assumption of risk neutrality in the HJM model leads to:

$$U(t, T, r_t) + (1/2)V^2(t, T) = r_t.$$

Equivalently,

$$r_t - \int_t^T \alpha(t, s)\mathrm{d}s + (1/2)\left(\int_t^T \varphi(t, s)\mathrm{d}s\right)^2 = r_t,$$

which implies

$$\int_t^T \alpha(t, s)\mathrm{d}s = (1/2)\left(\int_t^T \varphi(t, s)\mathrm{d}s\right)^2. \qquad (66)$$

After deriving the previous expression with respect to T, it is obtained that

$$\alpha(t, T) = \varphi(t, T)\int_t^T \varphi(t, s)\mathrm{d}s. \qquad (67)$$

Note that the forward rate volatility integral is the bond price volatility as in equation (38). Now then, on the basis of (28), it can be written

$$\sigma(t, T) = \int_t^T \varphi(t, s)\mathrm{d}s = -V(t, T),$$

then

$$-\sigma(t, T) = -\frac{\alpha(t, T)}{\varphi(t, T)} = V(t, T).$$

Therefore, the bond price equation defined in (56), under the assumption of risk neutrality, becomes

$$\begin{aligned}\mathrm{d}B(t, T) &= r_t B(t, T)\mathrm{d}t - \left(\int_t^T \varphi(t, s)\mathrm{d}s\right)B(t, T)\mathrm{d}W_t \\ &= r_t B(t, T)\mathrm{d}t - \frac{\alpha(t, T)}{\varphi(t, T)}B(t, T)\mathrm{d}W_t. \\ &= r_t B(t, T)\mathrm{d}t - \sigma(t, T)B(t, T)\mathrm{d}W_t. \qquad (68)\end{aligned}$$

[5] The correct thing to do is to say that there is an equivalent martingale measure.

The previous equation's deterministic component implies exponential growth in the bond price with a trend equal to the short rate. In this way, the stochastic differential equation that drives the dynamics of the instantaneous forward rate, under the assumption of risk neutrality, now takes the form:

$$\mathrm{d}f(t,T) = \left(\varphi(t,T)\int_t^T \varphi(t,s)\mathrm{d}s\right)\mathrm{d}t + \varphi(t,T)\mathrm{d}W_t. \tag{69}$$

Consequently, the forward rate trend is implicitly calibrated based on its volatility $\varphi(t,T)$.

3.5 Alternative representations of forward and short rates

It is common to find other representations of forward and short rates in the HJM methodology in literature. If it is denoted

$$V(t,T) = -\int_t^T \varphi(t,s)\mathrm{d}s = -\sigma(t,T),$$

and using (58) and (61), it is found that

$$\mathrm{d}f(t,T) = V_T(t,T)V(t,T)\mathrm{d}t - V_T(t,T)\mathrm{d}W_t \tag{70}$$

and

$$r_t = f(0,t) + \int_0^t V_t(s,t)V(s,t)\mathrm{d}s - \int_0^t V_t(s,t)\mathrm{d}W_s. \tag{71}$$

The application of Leibniz's rule and the fact that $V(t,t) = 0$ lead to:

$$\mathrm{d}r_t = f_t(0,t)\mathrm{d}t + \left(\int_0^t [V_{tt}(s,t)V(s,t) + V_t(s,t)^2]\mathrm{d}s\right)\mathrm{d}t - \left(\int_0^t V_{tt}(s,t)\mathrm{d}W_s\right)\mathrm{d}t - V_t(t,t)\mathrm{d}W_t$$

Finally, notice that the price of the bond satisfies

$$\mathrm{d}B(t,T) = r_t B(t,T)\mathrm{d}t + V(t,T)B(t,T)\mathrm{d}W_t,$$

equivalently

$$\mathrm{d}B(t,T) = r_t B(t,T)\mathrm{d}t - \sigma(t,T)B(t,T)\mathrm{d}W_t. \tag{72}$$

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3.6 A first example of the HJM methodology

This section illustrates the HJM methodology through a simple example. Assume that $\varphi(t,T)=\varphi_0=$ constant. Note, first, that based on (67), it follows that

$$\alpha(t,T)=\varphi(t,T)\int_t^T \varphi(t,s)\mathrm{d}s=\varphi_0^2(T-t).$$

Under (70), it is satisfied that

$$\begin{aligned} f(t,T)&=f(0,T)+\varphi_0^2\int_0^t (T-s)\mathrm{d}s+\varphi_0 W_t \\ &=f(0,T)+\varphi_0^2 t(T-(1/2)t)+\varphi_0 W_t. \end{aligned}$$

Thus,

$$r_t=f(0,t)+(1/2)\varphi_0^2 t^2+\varphi_0 W_t.$$

In the same way, equation (68) leads to

$$\begin{aligned} \mathrm{d}B(t,T)&=r_t B(t,T)\mathrm{d}t-\frac{\alpha(t,T)}{\varphi(t,T)}B(t,T)\mathrm{d}W_t \\ &=r_t B(t,T)\mathrm{d}t-\varphi_0(T-t)B(t,T)\mathrm{d}W_t. \end{aligned}$$

On the other hand, (59) implies that

$$\begin{aligned} B(t,T)&=\exp\left\{-\int_t^T f(0,s)\mathrm{d}s-\int_t^T[\varphi_0^2 t(s-(1/2)t)+\varphi_0 W_t]\mathrm{d}s\right\} \\ &=\exp\left\{\int_0^t f(0,s)\mathrm{d}s-\int_0^T f(0,s)\mathrm{d}s-\int_t^T[\varphi_0^2 t(s-(1/2)t)+\varphi_0 W_t]\mathrm{d}s\right\} \\ &=\frac{B(0,T)}{B(0,t)}e^{-\frac{1}{2}\varphi_0^2 tT(T-t)-\varphi_0(T-t)W_t}. \end{aligned}$$

From the above, it is observed the dependence of the price $B(t,T)$ with an available yield curve at time t=0, $R(0,T)=-\ln B(0,T)/T$. Finally, it is important to note that $B(t,T)$ is a random variable due to the presence of W_t in the exponential term.

3.7 A second example of the HJM methodology

The importance of the HJM methodology is best understood through illustrative examples. With this purpose in mind, a further example is developed.

Suppose interest rates are described by a HJM model with single source of uncertainty and forward rate volatilities

$$\varphi(t,T) = \varphi_0 e^{-\kappa(T-t)}.$$

as in the Vasicek model. Under equation (65), it follows that

$$\alpha(t,T) = \varphi(t,T)\int_t^T \varphi(t,s)\mathrm{d}s = \frac{\varphi_0^2}{\kappa}\left(e^{-\kappa(T-t)} - e^{-2\kappa(T-t)}\right)$$

Evaluating the integrals in (37) yields

$$f(t,T) = f(0,T) + \frac{\varphi_0^2}{2\kappa^2}\left((1-e^{-\kappa T})^2 - (1-e^{-\kappa(T-t)})^2\right) + \varphi_0 e^{-\kappa(T-t)} X^*(t) \tag{73}$$

where

$$X^*(t) = \int_0^t e^{-\kappa(t-\tau)}\,\mathrm{d}W^*(\tau) \tag{74}$$

is a normally distributed random variable under **P*** with zero mean and variance

$$\mathrm{Var}^* X^*(t) = \frac{1}{2\kappa}(1-\mathrm{e}^{-2\kappa t}) \tag{75}$$

The short rate is

$$r(t) = f(0,t) + \frac{\varphi_0^2}{2\kappa^2}(1-e^{-\kappa t})^2 + \varphi_0 X^*(t) \tag{76}$$

To express the future forward rate curves in terms of the short rate, substitute for $X^*(t)$ from (76) to (73). This yields

$$f(t,T) = f(0,T) + e^{-\kappa(T-t)}(r(t) - f(0,t)) + \frac{\varphi_0^2}{2\kappa^2} e^{-\kappa(T-t)}(1-e^{-\kappa(T-t)})(1-e^{-2\kappa t}) \tag{77}$$

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For a given value of $r(t)$, it is a deterministic function of T and t that is not supposed to be updated. It will nevertheless be replaced by the realized forward rate curve at time t.

The stochastic differential equation corresponding to (74) is

$$\mathrm{d}r = \kappa(\theta(t) - r)\,\mathrm{d}t + \varphi_0\,\mathrm{d}W^* \tag{78}$$

where

$$\theta(t) = f(0,t) + \frac{1}{\kappa}\frac{\partial}{\partial t} f(0,t) + \frac{\varphi_0^2}{2\kappa^2}(1 - e^{-2\kappa t}) \tag{79}$$

The Heath/Jarrow/Morton model assumes at time zero that $\theta(t)$ is a deterministic function for all $t \geq 0$. Yet it will be changed at subsequent repricing dates.

The same is true for the Hull/White and any other descriptive models.

Observe now that equation (68) leads to

$$\begin{aligned}\mathrm{d}B(t,T) &= r_t B(t,T)\mathrm{d}t - \frac{\alpha(t,T)}{\varphi(t,T)} B(t,T)\mathrm{d}W_t \\ &= r_t B(t,T)\mathrm{d}t - \frac{\varphi_0}{\kappa}\left(1 - e^{-\kappa(T-t)}\right)B(t,T)\mathrm{d}W_t,\end{aligned}$$

It follows from equations (5) and (73) that

$$\begin{aligned}B(t,T) &= \exp\left\{-\int_t^T \left[f(0,s) + \frac{\varphi_0^2}{2\kappa^2}\left[(1 - e^{-\kappa s})^2 - (1 - e^{-\kappa(s-t)})^2\right] + \varphi_0 \int_0^t e^{-\kappa(s-u)}\mathrm{d}W_u\right]\mathrm{d}s\right\} \\ &= \frac{B(0,T)}{B(0,t)}\exp\left\{-\frac{\varphi_0^2}{\kappa^3}\,(1 - e^{-\kappa t})\left(1 - e^{-\kappa(T-t)}\right) + \frac{\varphi_0^2}{4\kappa^3}\,(1 - e^{-2\kappa t})\left(1 - e^{-2\kappa(T-t)}\right)\right. \\ &\qquad \left. - \frac{\varphi_0}{\kappa}\left(1 - e^{-\kappa(T-t)}\right)X^*(t)\right\}.\end{aligned}$$

3.8 Dynamics of the forward rate with two risk factors

If a single factor is considered in the HJM model, the bonds of different maturities are perfectly correlated. This situation can be corrected by including other risk factors. Suppose that

$$\mathrm{d}f(t,T) = \alpha(t,T)\mathrm{d}t + \varphi_1(t,T)\mathrm{d}W_{1t} + \varphi_2(t,T)\mathrm{d}W_{2t}.$$

where $\varphi_1(t,T) = \varphi_{01}$ and $\varphi_2(t,T) = \varphi_{02}e^{-\kappa(T-t)}$. Also, suppose that

$$\mathrm{Cov}(\mathrm{d}W_{1t}, \mathrm{d}W_{2t}) = 0.$$

The term $\mathrm{d}W_{1t}$ is a long-term risk factor since it uniformly transfers the forward rate to all maturities. The $\mathrm{d}W_{2t}$ term affects the forward rate on short maturities more than the long-term factor. The results of the previous two sections lead to

$$f(t,T)=f(0,T)+\varphi_{01}^2 t(T-(1/2)t)+\varphi_{01}W_{1t}+\frac{\varphi_{02}^2}{2\kappa^2}+[2e^{-\kappa T}(e^{\kappa t}-1)-e^{-2\kappa T}(e^{2\kappa t}-1)]$$
$$+\varphi_{02}\int_0^t e^{-\kappa(T-s)}\mathrm{d}W_{2s}.$$

Thus, $r_t=f(0,t)+\frac{1}{2}\varphi_{01}^2 t^2+\varphi_{01}W_{1t}+\frac{\varphi_{02}^2}{2\kappa^2}(1-e^{-\kappa t})^2+\varphi_{02}\int_0^t e^{-\kappa(t-s)}\,\mathrm{d}W_{2s}$.

3.9 Discreet version of the HJM model

Below it is considered a discrete version of Heath, Jarrow and Morton (1990) model. The process of forward rates in periods of length Δ is examined instead of instantaneous forward rates. The quantities $\alpha_{i,j}$ and $\varphi_{i,j}$, $i,j=1,2,\ldots,k$, are defined as the trend and standard deviation, respectively, of the discretized process of the forward rate between times $j\Delta$ and $j\Delta+\Delta$ viewed at time $i\Delta$. That is, the discrete version of (58) is given by:

$$\mathrm{d}f(t,j\Delta,j\Delta+\Delta)=\alpha_{i,j}\mathrm{d}t+\varphi_{i,j}\mathrm{d}W_t,$$

when $t=i\Delta$. In this manner

$$V_{i,j}\approx\frac{V_{i,j+1}+V_{i,j}}{2}$$

and

$$\frac{\Delta V_{i,j}}{\Delta}\approx\frac{V_{i,j+1}-V_{i,j}}{\Delta}.$$

Therefore, it can be written, under (70), that

$$\alpha_{i,j}=V_{i,j}\frac{\Delta V_{i,j}}{\Delta}=\frac{V_{i,j+1}^2-V_{i,j}^2}{2\Delta}$$

and

$$\varphi_{i,j}=\frac{V_{i,j+1}-V_{i,j}}{\Delta},$$

where $V_{i,j}$ is the value $V(t,T)$ when $t=i\Delta$ and $T=j\Delta$. Now, since $V_{i,i}=0$, it follows from (66) that

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$$\sum_{j=1}^{k} \alpha_{i,j}\,\Delta = \frac{1}{2}\left(\sum_{j=1}^{k} \varphi_{i,j}\,\Delta\right)^2$$

Or

$$\sum_{j=1}^{k} \alpha_{i,j} = \frac{1}{2}\Delta\left(\sum_{j=1}^{k} \varphi_{i,j}\right)^2. \tag{80}$$

3.10 HJM Monte Carlo simulation

The Monte Carlo simulation method can be used to estimate the HJM model since $f(t,T)$ and $B(t,T)$ are random variables. The period over which the simulation runs is divided into n subintervals of the same length, Δ. Thus, the discrete version of (58) consists of

$$f_{i+1,j} - f_{i,j} = \alpha_{i,j}\Delta + \varphi_{i,j}\varepsilon\sqrt{\Delta}, \tag{81}$$

where $f_{i,j}$ denotes $f(i\Delta, j\Delta, j\Delta + \Delta)$, this is, $f_{i,j}$ is the forward rate between periods $j\Delta$ y $(j+1)\Delta$ at time $i\Delta$. The random variable ε is assumed to have a standard normal distribution. Values $\alpha_{i,j}$ can be calculated from the $\varphi_{i,j}$ values using (80). Likewise, at time $i\Delta$, the bonds' prices at maturity $j\Delta$ are stored for $i + 1 \leq j \leq n$. On the other hand, equation (72) becomes

$$\frac{B_{i+1,j} - B_{i,j}}{B_{i,j}} = \left(\frac{1 - B_{i,j}}{B_{i,j}}\right) + V_{i,j}\varepsilon\sqrt{\Delta}$$

Or

$$B_{i+1,j} = B_{i,j}\left(\frac{1}{B_{i,j}} + V_{i,j}\varepsilon\sqrt{\Delta}\right), \tag{82}$$

where $B_{i,j}$ is the price at time $i\Delta$ of a bond maturing at time $j\Delta$. The rate $R(t,T) = -\ln B(t,T)/(T-t)$ is then calculated.

4. Future research proposals

Based on the development of the previous sections, some of the areas that offer opportunities for research are presented below; the list is not intended to be exhaustive,

- Term structures with multiple factors and several sources of uncertainty modeled with Lévy processes combined with Cox processes and modulated by Markov chains.
- Quantum finance and term structures.
- Stochastic models of macroeconomic equilibrium that explain the behavior of the interest rate, abandoning the assumption of normality.
- Effects of negative interest rates on pension funds.
- Managing the risk of microcredit interest rates, with multiple factors, in the environment of the COVID-19 pandemic.
- Development of stochastic macroeconomic models with the health sector and its vulnerability to extreme events (catastrophes, pandemics -COVID-19-, etc.) that explain the future behavior of term structures.
- Effects of uncertainty in economic policy on term structures.
- Behavior of financial agents (behavioral finance) in debt markets and artificial intelligence.
- Applications of Data Science to the debt market.
- Effects of the COVID-19 pandemic on debt markets and its impact on term structures.

5. Conclusions

During the development of this work were reviewed the trends and perspectives of the theory of term structures. The paper carries out a detailed follow-up of the evolution of the theory of the term structure and the models that have been proposed in the literature. The provided list of future research proposals offers multiple opportunities for future work.

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